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A DYNAMIC INJECTION OPERATOR IN A MULTIGRID SOLUTION OF CONVECTION-DIFFUSION EQUATIONS[1]

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SUMMARY

A multigrid method is studied for the solution of a linear system resulting from the high-order nine-point discretization of the convection-diffusion equations. The residual injection operator is used as a substitute for the usual full-weighting in the multigrid process. A heuristic analysis is given to obtain a dynamic injection operator that is cost-effective for both diffusion- and convection-dominated problems. Numerical experiments are employed to test the stability and efficiency of the proposed method. © 1998 John Wiley & Sons, Ltd.

KEY WORDS: multigrid method; residual transfer; convection-diffusion equation

1. INTRODUCTION

The general convection-diffusion equation is considered:

$$\begin{aligned} &u_{xx} + u_{yy} + p(x, y)u_x + q(x, y)u_y = f(x, y), \quad (x, y) \in \Omega, \\ &\lambda u + \rho \frac{\partial u}{\partial n} = g(x, y), \quad (x, y) \in \partial\Omega, \end{aligned} \tag{1}$$

where λ, ρ are real numbers, $p(x, y)$ and $q(x, y)$ are functions of x and y. Ω is a convex domain and $\partial\Omega$ is the boundary of Ω. Let

$$\mathscr{P} = \max\left(\sup_{(x,y)\in\Omega} |p(x, y)|, \sup_{(x,y)\in\Omega} |q(x, y)| \right),$$

and define the cell-Reynolds number as

$$\mathscr{R} = \frac{\mathscr{P}h}{2},$$

where h is the uniform mesh width in the x- and y-directions. For $\mathscr{R} \leq 1$, Equation (1) is diffusion-dominated. Otherwise it is convection-dominated.

When $p(x, y)$ and $q(x, y)$ are oscillatory on Ω, the direction of the convection changes rapidly; in particular, when Equation (1) represents a recirculating flow problem and Ω contains stagnation points (turning points). The standard full-weighting usually misrepresents that characteristics of the flow around the turning points. By projecting residuals with

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misrepresented characteristics to the coarse grid, the coarse grid subproblem fails to approximate the one of the fine grid and causes instability on the fine grid for high-Reynolds flow.

Brandt and Yavneh [1] proposed a multigrid method (using a preconditioned five-point scheme) to accelerate the convergence of problems with high Reynolds recirculating flows. They utilized a dynamic overweighted residual (the coefficients of which are computed at each grid level) to reach this goal. Their method shows a very significant improvement in convergence rates at little cost.

Gupta *et al.* [2] utilized a fourth-order compact nine-point finite difference scheme (NPF) in the multigrid process. Using a red–black Gauss–Seidel smoother, they considered a residual injection operator (or scaled injection operator with the injection factor applied uniformly on all grid levels and taken as > 1.0) instead of the standard full-weighting operator to solve Equation (1) with high-cell-Reynolds numbers. The same approach was extended by Zhang [3] that implemented NPF with a four-color ordering. Their numerical results show that the residual injection operator always converges, whereas the full-weighting diverges when $\mathscr{R}$ is large.

For the diffusion-dominated problem, Gupta *et al.* derived a scaled injection operator (with the injection factor close to 0.54) that performs better than full-weighting and half-injection. However, this injection operator is not good for convection-dominated problems, particularly when the values of $\mathscr{R}$ are large.

The idea behind the methods of Gupta *et al.*, Zhang, and Brandt and Yavneh is that when $\mathscr{R}$ increases, the value of the injection factor (or overweighted residual) β (whether used uniformly or computed dynamically) should also increase up to a constant (may be problem dependent). Unfortunately, the optimum β obtained for large $\mathscr{R}$ cannot be considered for diffusion-dominated problems. Even if $\mathscr{R}$ is moderated, this optimum may lead to divergence. Using a nine-point high-order discretization formula and the red–black Gauss–Seidel (RBGS) smoother, a dynamic injection operator that alleviate these difficulties is derived. Instead of using a uniformly coefficient β, or computing one on all the grid levels, a residual injection factor is computed for each grid point. This dynamic factor is the function of the actual value of the local Reynolds number (defined below) at the grid point considered.

This paper is organized as follows. The multigrid algorithm is briefly presented in Section 2. To understand the analysis Section 3 starts with some basic computations on a constant coefficients problem. Simple assumptions are made and the results of the computations (on the constant coefficients problem) are used to design the dynamic injection operator. In Section 4 the performance of the new operator is tested on problems taken from the literature. Remarks and conclusions are given in Sections 5 and 6, respectively.

2. MULTIGRID IMPLEMENTATION

The multigrid solver is designed as follows:

1. Start from the fine grid by some initial guess and perform v_1 RBGS relaxation sweeps.
2. Calculate the residuals corresponding to the coarse grid points, multiply the residuals by a factor, β and inject the residuals to the coarse grid.
3. Perform μ multigrid cycles on this grid.
4. Interpolate the coarse grid correction to the fine grid by bilinear interpolation.
5. Perform v_2 RBGS relaxation sweeps on the fine grid.

v_1 and v_2 are the numbers of presmoothing and postsmoothing sweeps. The value $\mu = 2$ is taken, such that this algorithm corresponds to a W-cycle. For the relaxation technique, the red–black Gauss–Seidel (RBGS) smoother is considered.

The test problems given here are solved using a uniform mesh size h on a rectangular domain Ω. For the multigrid solver, standard coarsening technique (the mesh size of the coarse grid doubles that of the fine grid) is used.

The performance is measured by the number (n) of W-cycles needed to reach a given reduction (tol) of the l_2-norm of the residual, i.e. $\|r^{(n)}\|_2 < \text{tol} * \|r^{(0)}\|_2$, where $r^{(k)}$ denotes the residual after k W-cycles. All computations were done on a SGI (Silicon Graphic Indy) workstation using FORTRAN 77 programming language in double precision.

NPF was used to discretize Equation (1). Details of the discretization scheme can be found in Reference [4] and the multigrid implementation of this scheme can be found in Reference [2]. The reason for choosing this fourth-order discretization scheme is that it is stable for all ϵ [2], so the magnitude of the cell-Reynolds numbers that might cause divergence if a low-order, e.g. a standard five-point formula, were employed, is of no concern. In addition, NPF was used on all the grid levels. This choice was analytically shown to be stable in Reference [5].

3. RESIDUAL TRANSFER ANALYSIS

To find the optimal injection operator is to find the optimal scaling factor β to represent $\bar{r}_{i/2,j/2}$ (right-hand component of the coarse grid equation obtained from the transfer of the residual) in terms of $r_{i,j}$ (component of the fine grid residual to be transferred), as accurately as possible, i.e.

$$\bar{r}_{i/2,j/2} = \beta r_{i,j}. \tag{2}$$

When the problem is diffusion-dominated, Gupta *et al.* [2] were able to derive an injection factor ($\beta = 0.5467$)[2] the injection operator of which performs better than both the full-weighting and the half-injection operators. For convection-dominated problems, the standard full-weighting transfers a bad representation of the problem on the fine grid to the coarse grid. This may lead to slow convergence or even divergence. Of course, for convection-dominated problems the residual injection factor will generally change from the one obtained for diffusion-dominated problems. Since the smooth components of the errors increase as $\mathscr{R}$ increases to infinity, the scaling factor β may be increased to reflect this fact. Although there is no absolute guarantee that any single factor will work for all practical problems, in general the residual injection factor is indeed an increasing function of the cell-Reynolds number $\mathscr{R}$, and it approaches a constant when $\mathscr{R}$ tends to infinity. This constant may be problem-dependent, but it is usually between 1 and 2. Larger injection factors are unlikely to be useful in practice, if only because the corresponding amplification of non-smooth error components means that much better smoothing is then required [1].

Although the order of the injection operator is 0 (see Reference [6]) and the combination of injection with bilinear interpolation (order 2) violates the order rule set up in References [6,7] for small $\mathscr{R}$ (second-order equation), the numerical experiments in this study showed that the injection operator is better than full-weighting in terms of CPU time [8].

[2] This value was derived using the analysis of RBGS and full-weighting, and some geometric consideration when NPF is employed.

The goal in this section is to find a dynamic injection factor that can be used both for diffusion- and convection-dominated problems without deteriorating the rate of convergence. This analysis is introduced by a simple example.

3.1. A simple experiment

Consider the following constant coefficient problem:

Problem 1

$$-\epsilon(u_{xx}+u_{yy})-\frac{\sqrt{2}}{2}u_x-\frac{\sqrt{2}}{2}u_y=0 \quad \text{on } \Omega=(0,1)\times(0,1), \tag{3}$$

$$u=0 \quad \text{on } \partial\Omega, \tag{4}$$

where ϵ is positive. The exact solution is $u=0$.

The relationship between the injection factor β and the cell-Reynolds numbers $\mathscr{R}$ is required. For $h=1/64$, this problem is solved with NPF-MG. The number of iterations is reported for different values of ϵ and the injection factor β in Table I. β is uniform on all the grid levels.

From Table I, when $\mathscr{R}$ is small (ϵ is large) on the finest grid, the best β is 0.5467. But when $\mathscr{R}$ is large (ϵ small), as β increases, the rate of convergence improves. Large β that are good for high-cell-Reynolds numbers are useless for the problem of small-cell-Reynolds numbers. Even for moderated $\mathscr{R}$, large β may lead to divergence. From these computations, the difficulty for a particular value of ϵ to find the appropriate residual injection factor that gives the best convergence rate is noted. It can be stated, however, that when ϵ decreases (or $\mathscr{R}$ increase) then, β should increase.

In Table I, when $\epsilon=10^{-3}$ a deterioration of the convergence or even divergence for all the values of β is observed. A Fourier smoothing analysis was carried out using lexicographical and line Gauss–Seidel relaxations (it is important to note that it impossible to perform the same Fourier analysis with RBGS using a nine-point scheme). It was found that for a given mesh width h, the largest value of the smoothing number is obtained for $\epsilon=10^{-3}$ and this value increases with h. This observation suggests that some of the coarsest grid in the multigrid calculations needs to be removed.

Table I. Number of iterations for test Problem 1

ϵ	$\mathscr{R}$	β									
		0.5467	0.6	0.7	0.8	0.9	1.0	1.1	1.25	1.4	1.5
10^{1}	0.000552	4	4	7	9	12	16	48	div	div	div
10^{0}	0.005524	4	4	7	9	12	16	49	div	div	div
10^{-1}	0.055242	4	5	7	9	12	18	42	div	div	div
10^{-2}	0.552427	7	7	10	16	25	35	61	div	div	div
10^{-3}	5.52427	37	49	229	div	div	div	div	div	div	div
10^{-4}	55.2427	73	64	52	43	36	31	27	22	18	17
10^{-5}	552.427	110	93	69	53	42	34	28	22	18	15
10^{-6}	5524.27	110	92	68	52	41	33	27	21	17	15
10^{-7}	55242.7	110	92	68	52	41	33	27	21	17	15

$h=1/64$, $\text{tol}=10^{-5}$, 'div' stands for divergence.

Table II. Number of iterations for test Problem 1 when some coarsest grids were removed

ϵ	β									
	0.5467	0.6	0.7	0.8	0.9	1.0	1.1	1.25	1.4	1.5
10^{-2}	9	9	13	18	27	38	58	div	div	div
10^{-3}	31	32	35	41	53	81	194	div	div	div
10^{-4}	198	183	159	139	123	110	99	85	74	68

$h = 1/64$, $\text{tol} = 10^{-5}$, 'div' stands for divergence.

The same multigrid experiments were performed by removing some coarsest grid ($h = 1/2$ and $h = 1/4$). The results of Table II show an improvement of the convergence (or even the regain of convergence) for $\epsilon = 10^{-3}$. But when $\epsilon = 10^{-2}$ (or greater) and $\epsilon = 10^{-4}$ (or less) the convergence deteriorates.

It is important to note that for this constant coefficient problem, the full-weighting always leads to convergence. But for convection-dominated problems the number of iterations with the full-weighting is at least twice that obtained with the residual injection operator (with an appropriate factor) [8]. For the variable coefficient problems that are introduced later, full-weighting does not converge when $\mathscr{R} \gg 1$.

3.2. Dynamic injection

This section begins with some observations (on the coefficient β) based on the results obtained in References [1,2] and the ones just presented.

Observation 1. For the nine-point formula, the factor $\beta = 0.5467$ is very effective when $p = q = 0$ (and for diffusion-dominated problems).

$$\beta(\mathscr{R}) \approx \beta^{\min} = 0.5467 \quad \text{for } \mathscr{R} \approx 0.$$

Observation 2. The injection factor β is a bounded, increasing and continuous function of the cell-Reynolds numbers $\mathscr{R}$, with

$$\frac{\mathrm{d}\beta(\mathscr{R})}{\mathrm{d}\mathscr{R}} > 0, \qquad \text{for } 0 \le \mathscr{R} < v,$$

$$\frac{\mathrm{d}\beta(\mathscr{R})}{\mathrm{d}\mathscr{R}} = 0, \qquad \text{for } \mathscr{R} \ge v,$$

$$\beta(\mathscr{R}) = \beta^{\max}, \qquad \text{for } \mathscr{R} \ge v.$$

For a particular problem, it is difficult to find the value β that will give the best convergence rate for high-cell-Reynolds numbers. Even if it were possible, another problem would require a different β. Furthermore, in general, it will not be possible to compute $\mathscr{R}$ for variable coefficient problems, and therefore, to find the appropriate β. Instead, each grid point must be computed at an injection coefficient (that will be used to scale the residual) that reflects the actual values of the functions $p(x, y)$ and $q(x, y)$ (with respect to the mesh size h) at this point. First some assumptions are made.

Assumption 1. The residual scaling factor β is a piecewise linear function of the cell-Reynolds numbers.

Assumption 2. For any problem,

$$\beta(0) = 0.5467 \quad \text{and} \quad \beta(\mathscr{R}) = \beta^{\max} \approx 1.5 \quad \text{for } \mathscr{R} \geq v.$$

Assumption 3. RBGS is an acceptable smoother (smoothing number less than one) for NPF.

The first assumption is very crude but helps simplify the analysis. The results of Table I seem to suggest that the linear choice is not the best one (see the best β for $\epsilon \geq 10^{-3}$). But numerical experiments show that this choice turns out to be accurate.

The second assumption is based on the experiments and the heuristic analysis developed in Reference [2]. For small values of $\mathscr{R}$, $\beta = 0.5467$. For large $\mathscr{R}$, β may be problem dependent. In general, for recirculating flow problems with stagnation points, it is between one and two, therefore, the average of the two values, 1.5, is taken. The goal is not to use the value 1.5 uniformly on all grid levels, but to use it in designing the dynamic injection operator.

The last assumption is somehow related to the second one, in the sense that if RBGS is not a good smoother, large scaling factors (>2) are needed to get an acceptable rate of convergence. Instead of taking a large β with RBGS, the best thing is to choose another smoother that will require a smaller β. In addition, it is important to mention that it is not possible to carry out a Fourier analysis of NPF when RBGS is employed, since the red points can not be decoupled from the black points. However, in Reference [8], for constant coefficient problems, Gupta *et al.* compared the performances of RBGS with other relaxation techniques that were shown to have the smoothing property (smoothing number <1). They found that RBGS outperforms all these relaxation techniques.

Now, $\beta^{\min}$ will refer to 0.5467 and $\beta^{\max}$ to 1.5.

Consider the multigrid algorithm with $l+1$ levels: $0, \ldots, l$ and uniform square meshes on each level with mesh widths h_0 and $h_k = h_{k-1}/2$ for $k = 1, \ldots, l$. Let Ω_k, $k = 0, \ldots, l$ be the corresponding grid decompositions of the domain Ω. For a grid point (x, y) in Ω_k define the local Reynolds number (LR_k) by:

$$LR_k(x, y) = \frac{h_k}{2} \max(|p(x, y)|, |q(x, y)|).$$

It is easy to check that

$$\max_{(x,y)\in\Omega_k} LR_k(x, y) \leq \mathscr{R},$$

with equality, if $\mathscr{P}$ (defined in Section 1) is obtained on a grid point. If the functions p and q are constant, LR_k is constant on the level k. LR_k and $\mathscr{R}$ will therefore be equal.

At a grid point of Ω_k that is also in Ω_{k+1}, the corresponding residual must be scaled by a factor β that reflects the actual value of LR_k at (x, y). When this value is small, β should be close to $\beta^{\min}$, when the value is large, β should be closed to $\beta^{\max}$ obtained for large-cell-Reynolds numbers, and finally, if $LR_k(x, y)$ is moderated, the factor β should be a value between the previous two. According to the preceding assumptions, if $LR_k(x, y) = 0$ then $\beta = \beta^{\min}$ and if $LR_k(x, y) \approx \infty$, then $\beta = \beta^{\max}$.

Let $\tilde{v}$ be the minimum value of the function LR that requires the factor $\beta^{\max}$. Suppose we want to transfer the residual r (at a grid point (x, y)) from the grid level k to the grid level $k+1$. It is assumed that (x, y) is a grid point of both Ω_k and Ω_{k+1}. The following formula is used:

$$\bar{r}(x, y) = \beta_{di}(LR_k(x, y)) \times r(x, y), \tag{5}$$

where (x, y) is grid point of both Ω_k and Ω_{k+1}, and

$$\beta_{di}(LR_k(x,y)) = \begin{cases} \frac{\beta^{\max}-\beta^{\min}}{\tilde{\nu}} LR_k(x,y) + \beta^{\min} & \text{if} \quad LR_k(x,y) < \tilde{\nu}, \\ \beta^{\max} & \text{if} \quad LR_k(x,y) \geq \tilde{\nu}. \end{cases}$$

In Figure 1 β_{di} is plotted as a function of LR. It is important to note that the new residual injection factor acts locally. Each component of the residual to be injected on the coarse grid is treated individually. The components that have large LR will be scaled with high factor, whereas those with small LR will be scaled with low factor. For constant coefficient problems, β_{di} is constant for a given grid level k since LR_k is constant. The method used in this particular case will be somehow similar to the one of Brandt and Yavneh [1].

In this definition of the function β_{di}, the constant $\tilde{\nu}$ plays a major role. How could the appropriate value of $\tilde{\nu}$ be chosen? A small value of $\tilde{\nu}$ ($\beta_{di} \approx \beta^{\max}$) is good for high $\mathscr{R}$, but cannot be used when the problem is diffusion-dominated because divergence is more likely to be obtained. However, a large $\tilde{\nu}$ ($\beta_{di} \approx \beta^{\min}$) is best for small $\mathscr{R}$; even though it is not interesting for large $\mathscr{R}$, it is more likely to give convergence. A trade-off between these two cases must then be found, for one choice or the other will at least penalize moderated $\mathscr{R}$.

To introduce the function β_{di} in our multigrid process, Step 2 of the algorithm introduced above must be changed. The new Step 2 reads calculate the residuals corresponding to the coarse grid points and the values of β_{di}, multiply the residuals by a factor β_{di} and inject the residuals to the coarse grid.

This new algorithm does not introduce more computational complexity compared with the previous one. To carry out Step 2, a simple comparison to find $LR_k(x,y)$ is added to the old one (at all the grid levels the values of $p(x,y)$ and $q(x,y)$ have been precomputed during the initialization process) and $\beta_{di}(LR_k(x,y))$ is computed. The additional CPU time resulting from these changes is negligible.

4. NUMERICAL EXPERIMENTS

Here the performances of the new injection operator are tested. For the computations, $\tilde{\nu} = 1500.0$ is chosen. It can be shown that convergence for any values of $\mathscr{R}$ can be reached.

First, the results for Problem 1, introduced in Section 3.1, are given. The number of iterations is shown in Table III when $h = 1/32,\ 1/64,\ 1/128$. The convergence is always obtained. The comparison of the results in Tables I and III (column 3) shows that the dynamic

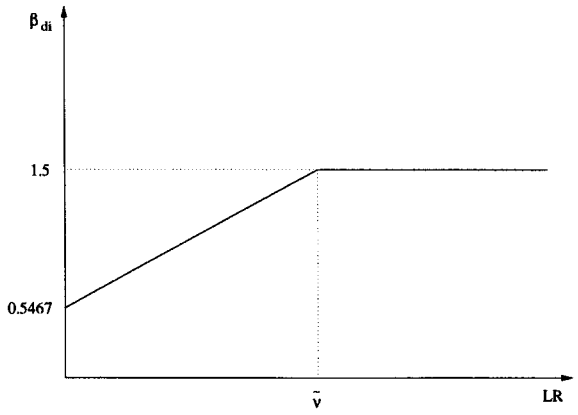

Figure 1. β_{di} as a function of LR.

Table III. Number of iterations for test Problem 1 with the dynamic injection factor β_{di}

ϵ	$h = 1/32$	$h = 1/64$	$h = 1/128$
10^1	4	4	4
10^0	4	4	4
10^{-1}	4	4	4
10^{-2}	10	7	5
10^{-3}	27	38	24
10^{-4}	48	64	91
10^{-5}	18	28	41
10^{-6}	15	15	13
10^{-7}	15	15	13

$\text{tol} = 10^{-5}$.

injection operator works very well when $\mathscr{R}$ is small or large. For moderated $\mathscr{R}$ ($\epsilon = 10^{-4}$, 10^{-5}) the dynamic injection is not optimum. This observation also appears in the results of the next problems. This difficulty can be easily overcome by choosing another value of $\tilde{v}$ (an example is given at the end of this section).

The following variable coefficient problems are introduced.

Problem 2

$$
\begin{aligned}
&-\epsilon(u_{xx}+u_{yy})+b_1(x,y)u_x+b_2(x,y)u_y=0 \quad \text{on } \Omega=(-1,1)\times(0,1),\\
&u|_{\partial_1\Omega}=1+\tanh(10+20x), \quad -1\le x\le 0, \qquad \left.\frac{\partial u}{\partial n}\right|_{\partial_2\Omega}=\left.\frac{\partial u}{\partial n}\right|_{\partial_3\Omega}=\left.\frac{\partial u}{\partial n}\right|_{\partial_4\Omega}=\left.\frac{\partial u}{\partial n}\right|_{\partial_5\Omega}=0,\\
&b_1(x,y)=y(1-x^2), \qquad b_2(x,y)=-x(1-y^2).
\end{aligned} \tag{6}
$$

Problem 3

$$
\begin{aligned}
&-\epsilon(u_{xx}+u_{yy})+b_1(x,y)u_x+b_2(x,y)u_y=f(x,y) \quad \text{on } \Omega=(0,0)\times(0,1),\\
&u(x,y)=0 \quad \text{on } \partial\Omega, \qquad b_1(x,y)=\sin(\pi y)\cos(\pi x), \qquad b_2(x,y)=-\cos(\pi y)\sin(\pi x),
\end{aligned} \tag{7}
$$

where ϵ is a positive number. The exact solution u on $\bar{\Omega}$ is $u = 0$ (Figure 2). Problem 2 was used by de Zeeuw and van Asselt [9] to prove the stability of some five-point star schemes with artificial viscosity. Problem 3, the characteristics of which form a single clockwise-rotating vortex, presents a stagnation point at (1/2, 1/2) (Figure 3). It was introduced by Brandt and Yavneh [1] in order to test the performances of their overweighted residual technique used to solve problems with recirculating flows.

For these two problems, the performance of the dynamic injection operator is tested. The problems for the injection factor constant on all the grid level $\beta = 0.5467$ and 1.0, respectively, are solved. The dynamic injection factor is also used. The results are summarized in Table IV for Problem 2 and in Table V for Problem 3.

The dynamic injection operator performs well for small and large values of ϵ. It does not deteriorate the rate of convergence. When $\beta = 0.5467$ uniformly on all the grid levels, there is always convergence but when ϵ decreases, its performance deteriorates. The constant $\beta = 1.0$ alleviates this difficulty but its utilization may sometimes lead to divergence.

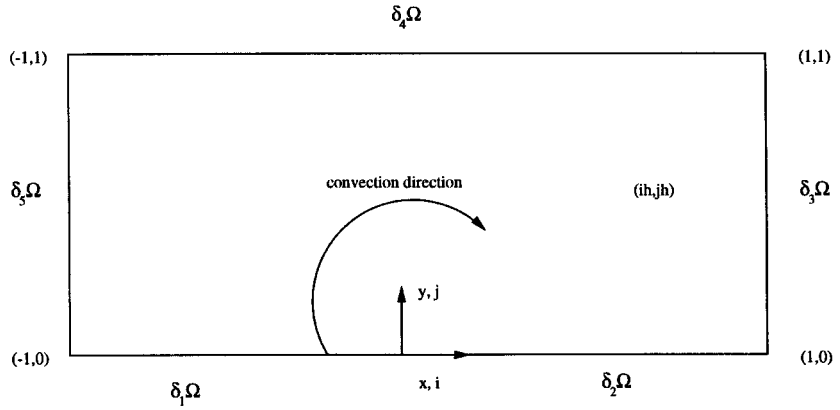

Figure 2. The domain Ω for Problem 2.

As seen in the constant coefficients problem, for $\epsilon = 10^{-4}$, 10^{-5} the dynamic injection displays some difficulties. The same experiments were carried out with different values of $\tilde{v}$. The best $\tilde{v}$ (for any values of ϵ) is ≈ 250 for Problems 1 and 3 and ≈ 2500 for Problem 2.

In the previous computations it was shown that for small $\mathscr{R}$, the dynamic injection operator performs as well as the constant injection operator, and when in the latter, the optimum coefficient ($\beta = 0.5467$) is uniformly applied at all the grid levels. Assume now that when $\mathscr{R}$ is large, the optimum β (the one that gives the best convergence rate) is known for a given problem; could the dynamic injection operator still have comparable results?

For Problem 3, $h = 1/128$, and different values of residual injection factor β (uniformly on all grid levels) and the dynamic injection operator (with $\tilde{v} = 250$), when ϵ varies from 10^{-4} to 10^{-9}, are applied. For each case the number of iterations is computed. The results are summarized in Table VI.

When ϵ vanishes, the optimum β is a number between 1.2 and 1.4. In fact, $\beta = 1.4$ is the optimum one. If we take a smaller tol the best convergence rate will be with this factor. This $\beta = 1.4$ agrees with the one found by Brandt and Yavneh [1].

Our dynamic operator and the optimum β have the same convergence rate. It is important to note that the utilization of the factor $\beta = 1.5$ deteriorates the convergence. In the dynamic

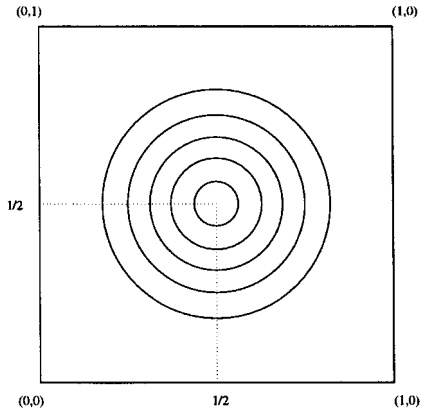

Figure 3. The domain Ω for Problem 3.

Table IV. Number of iterations for Problem 2

ϵ	β								
	$h = 1/32$			$h = 1/64$			$h = 1/128$		
	0.5467	1.0	β_{di}	0.5467	1.0	β_{di}	0.5467	1.0	β_{di}
10^{1}	4	12	4	4	12	4	4	14	4
10^{0}	4	12	4	4	12	4	4	14	4
10^{-1}	4	12	4	4	13	4	4	14	4
10^{-2}	5	29	5	4	48	4	4	38	4
10^{-3}	21	div	21	15	div	15	10	div	10
10^{-4}	64	40	56	70	52	59	67	div	79
10^{-5}	75	45	38	111	46	33	137	42	69
10^{-6}	75	45	38	112	45	30	156	44	24
10^{-7}	75	45	38	112	45	30	156	44	24

tol $= 10^{-4}$, 'div' stands for divergence.

operator, some components of the residual are scaled by 1.5 before they are injected to the next coarse grid. This does not affect the convergence of the multigrid algorithm.

5. REMARKS

In the preceding computations, a convergence criteria tol was chosen, which can be considered as weak. The use of a stronger criteria will lead to the same result but with larger number of iterations. Our goal was to show the efficiency of the new injection operator. The choice of the weak convergence criteria was determined by the NPF used for discretization of Equation (1). This scheme, though stable, displays slow convergence for large-cell-Reynolds numbers. This difficulty can easily be overcome by using, e.g. the minimal residual acceleration technique (that can reduce the number of iterations by $> 85\%$) [10].

Table V. Number of iterations for Problem 3

ϵ	β								
	$h = 1/32$			$h = 1/64$			$h = 1/128$		
	0.5467	1.0	β_{di}	0.5467	1.0	β_{di}	0.5467	1.0	β_{di}
10^{1}	3	11	3	3	12	3	3	12	3
10^{0}	3	11	3	3	12	3	3	12	3
10^{-1}	3	12	3	3	13	3	3	12	3
10^{-2}	6	15	6	5	21	5	4	22	4
10^{-3}	12	10	12	13	div	13	11	div	11
10^{-4}	46	29	44	59	25	56	41	29	39
10^{-5}	50	32	29	125	50	59	145	25	34
10^{-6}	50	32	24	127	50	34	112	24	22
10^{-7}	50	32	24	127	50	33	112	24	21

tol $= 10^{-4}$, 'div' stands for divergence.

Table VI. Number of iterations for Problem 3

ϵ	β						
	1.0	1.1	1.2	1.3	1.4	1.5	β_{di}
10^{-4}	29	64	div	div	div	div	27
10^{-5}	25	23	22	22	22	29	24
10^{-6}	24	22	21	21	21	36	21
10^{-7}	24	22	21	21	21	36	21
10^{-8}	24	22	21	21	21	36	21
10^{-9}	24	22	21	21	21	36	21

$h = 1/128$, tol $= 10^{-4}$, 'div' stands for divergence.

The choice of the values $\beta^{\max} = 1.5$ and $\tilde{\nu} = 1500$ can be seen arbitrary. The purpose was to use fixed values that can work well on all the test problems. There is an interaction between $\beta^{\max}$ and $\tilde{\nu}$. To have satisfactory results, larger $\beta^{\max}$ (or smaller) would require larger $\tilde{\nu}$ (or smaller). For instance, if $\beta^{\max} = 2$, $\tilde{\nu} > 10^6$ was found to be effective. This choice did not affect diffusion-dominated problems but was not as good as the one presented for the convection-dominated problems.

For a fixed $\beta^{\max}$, to find an effective $\tilde{\nu}$, convection-dominated problems with high $\mathscr{R}$ must be studied, since this is where the difficulties lie. On all the problems the values $\epsilon = 10^{-6}$ and $h = 1/64$ were fixed and started with $\tilde{\nu} = 1000$ until (by decreasing or increasing) the appropriate $\tilde{\nu}$ was obtained. This procedure of finding $\tilde{\nu}$ is computationally less difficult than obtaining the best β (since the best β depends on a particular problem with a given $\mathscr{R}$, whereas $\tilde{\nu}$ works across problems with any $\mathscr{R}$).

For Problem 3, when the physical viscosity coefficient ϵ vanishes, the coefficients of the equation are very small at and near the stagnation point (1/2, 1/2). In a neighborhood of this point, the equation behaves almost like a Poisson equation. The dynamic injection operator takes this fact into account, because the residual injection factor β at grid point in this neighborhood is closed to $\beta^{\min}$. The maximum error occurs near the stagnation point and the accuracy is slightly better with the dynamic operator. But this result is not significant enough to draw some general conclusions. More investigations need to be made in this direction.

The scaled injection operator (or in this work the dynamic injection operator), is used to find three objectives:

1. For convection-dominated problems, convergence must be recovered when standard multigrid techniques diverge [2].
2. The use of the operator is required as an acceleration technique for both diffusion and convection-dominated problems [8,2].
3. The design of stable and efficient multigrid solvers is required [8,2,5,3]. By efficiency, this means building convergent and simple multigrid algorithms with no additional computational cost.

The last objective is different from the one of algebraic multigrid solvers, the goal of which is to build robust solvers. For instance, the matrix-dependent prolongations and restrictions technique [11] leads to a better convergence rate. In this approach, incomplete line relaxations and sophisticated inter-grid transfer operators are employed at the expense of an increase of memory usage and computational work.

6. CONCLUSION

In this article, the attention has been focused on designing a dynamic injection operator that can be used both for diffusion- and convection-dominated problems. This operator does not introduce more computational cost in the multigrid process and works very well for any value of the cell-Reynolds numbers (whether small, moderated or very large).

The utilization of this new operator prevent the definition of an optimal injection factor (depending on problems) for a given value of the cell-Reynolds number. A unique dynamic operator can be used efficiently for a variety of problems but by a simple change of parameters, one can derive a dynamic residual injection operator that is optimal for a particular problem.

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